Section 2.2

Grade: «grade»
Subject: «subject»
Date: «date»

Limits Involving Infinity Exploration

Equation

NORMAL FLOAT AUTO REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = \frac{X^3+2X+3}{4X^3-4}$

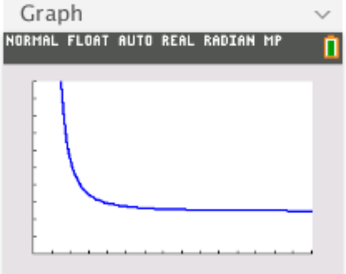
$Y_2 = \frac{3X^2-X+5}{X^2-4}$

$Y_3 = \frac{1}{3-X}$

$Y_4 =$

$Y_5 =$

Graph



Table

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y_1
-.003	-.7485
-.002	-.749
-.001	-.7495
0	-.75
.001	-.7505
.002	-.751
.003	-.7515
.004	-.752
.005	-.7525
.006	-.753
.007	-.7535

X=-.003

$$\frac{\cancel{x}^3}{4\cancel{x}^3} = \frac{1}{4}$$

Equation

NORMAL FLOAT AUTO REAL RADIAN MP

Plot1 Plot2 Plot3

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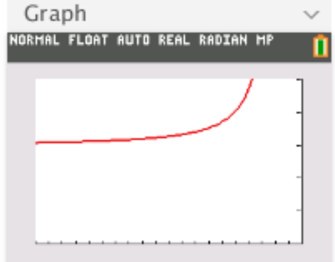


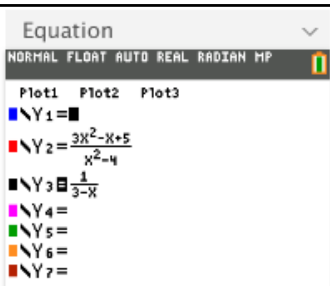
Table

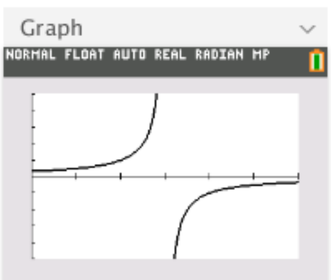
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

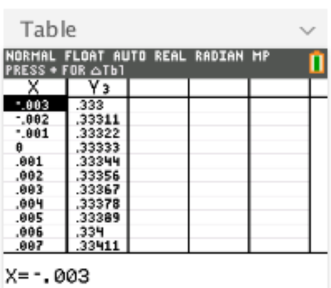
X	Y_2
-.003	-1.251
-.002	-1.251
-.001	-1.25
0	-1.25
.001	-1.25
.002	-1.25
.003	-1.249
.004	-1.249
.005	-1.249
.006	-1.249
.007	-1.248

X=-.003

$$\frac{3\cancel{x}^2}{\cancel{x}^2} = 3$$







X	Y_3
-.003	.333
-.002	.33311
-.001	.33322
0	.33333
.001	.33344
.002	.33356
.003	.33367
.004	.33378
.005	.33389
.006	.334
.007	.33411

$$\lim_{x \to 3^-} \frac{1}{3-x} = \infty$$

$$\lim_{x \to 3^+} \frac{1}{3-x} = -\infty$$

$$\lim_{x \to 3} \frac{1}{3-x} \quad \text{DNE}$$

vertical asymptote
$x = 3$

$$\lim_{x \to \infty} \frac{1}{3-x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{3-x} = 0$$

Finite Limits as $x \to \pm\infty$

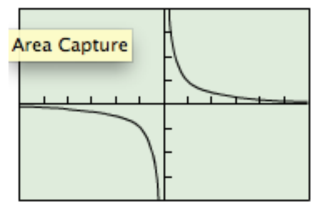
The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, when we say "the limit of f as x approaches infinity" we mean the limit of f as x moves increasingly far to the right on the number line. When we say "the limit of f as x approaches negative infinity ($-\infty$)" we mean the limit of f as x moves increasingly far to the left. (The limit in each case may or may not exist.)

(a) as $x \to \infty$, $(1/x) \to 0$ and we write

$$\lim_{x \to \infty} (1/x) = 0,$$

(b) as $x \to -\infty$, $(1/x) \to 0$ and we write

$$\lim_{x \to -\infty} (1/x) = 0.$$



[–6, 6] by [–4, 4]

Figure 2.9 The graph of $f(x) = 1/x$.

We say that the line $y = 0$ is a *horizontal asymptote* of the graph of f.

DEFINITION Horizontal Asymptote

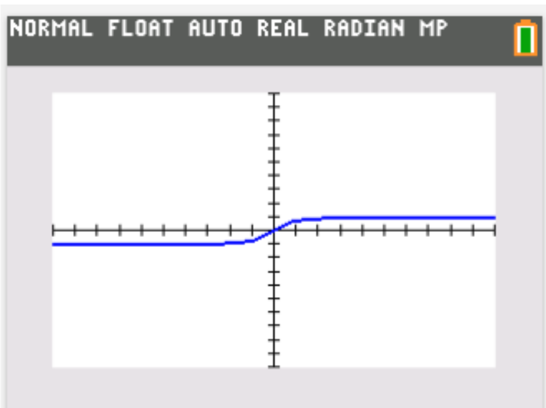
The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

EXAMPLE 1 Looking for Horizontal Asymptotes

Use graphs and tables to find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, and identify all horizontal asymptotes of $f(x) = x/\sqrt{x^2+1}$.

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

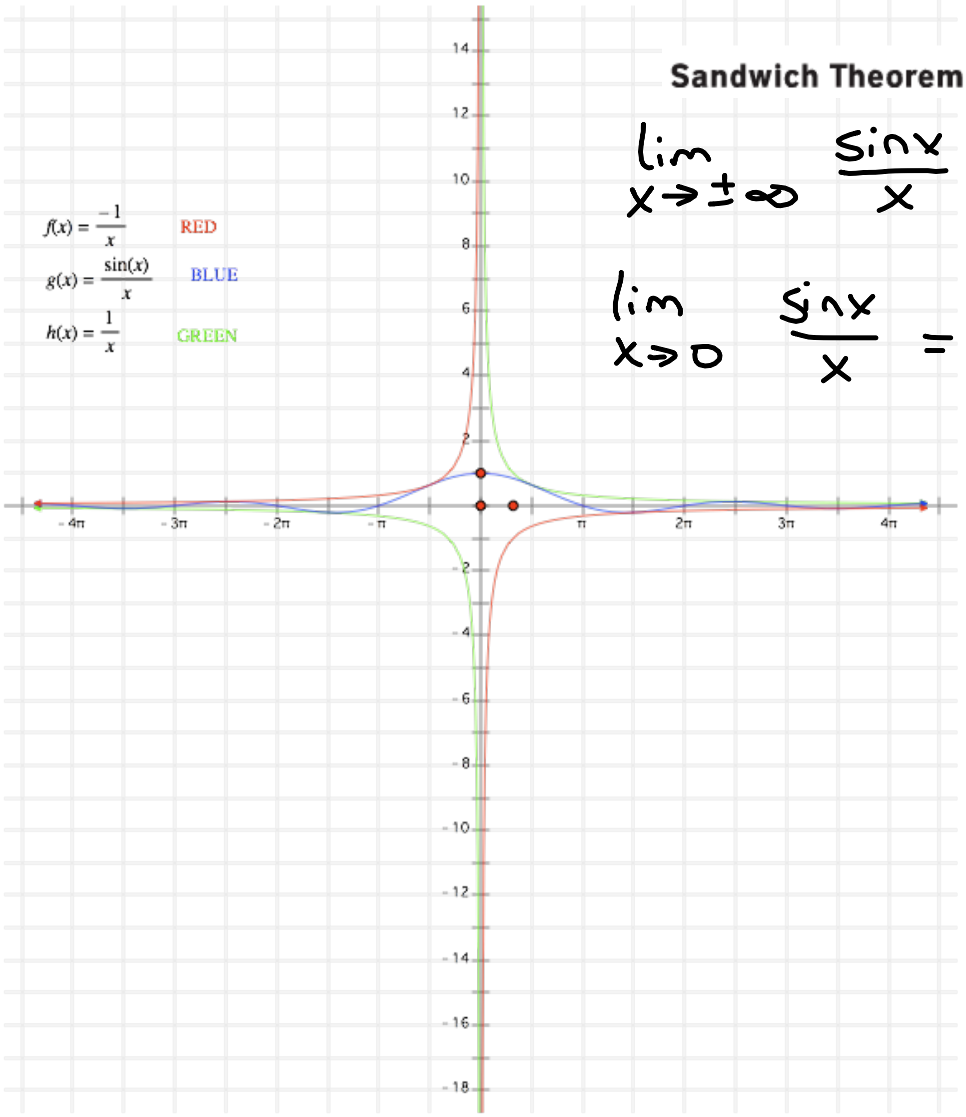


$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$$

EXAMPLE 2 Finding a Limit as x Approaches ∞

Find $\lim_{x\to\infty} f(x)$ for $f(x) = \frac{\sin x}{x}$.



Sandwich Theorem Revisited

$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

THEOREM 5 Properties of Limits as $x \to \pm\infty$

If L, M, and k are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$
6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

August 31, 2015

EXAMPLE 3 Using Theorem 5

Find $\lim_{x\to\infty} \frac{5x + \sin x}{x}$.

SOLUTION

Notice that

$$\frac{5x + \sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x}.$$

So,

$$\lim_{x\to\infty} \frac{5x + \sin x}{x} = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{\sin x}{x} \quad \text{Sum Rule}$$

$$= 5 + 0 = 5. \quad \text{Known Values}$$

Now try Exercise 25.

EXPLORATION 1 Exploring Theorem 5

We must be careful how we apply Theorem 5.

1. (Example 3 again) Let $f(x) = 5x + \sin x$ and $g(x) = x$. Do the limits as $x\to\infty$ of f and g exist? Can we apply the Quotient Rule to $\lim_{x\to\infty} f(x)/g(x)$? Explain. Does the limit of the quotient exist?
2. Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Describe the behavior of f and g as $x\to\infty$. Can we apply the Sum Rule to $\lim_{x\to\infty} (f(x) + g(x))$? Explain. Does the limit of the sum exist?
3. Let $f(x) = \ln(2x)$ and $g(x) = \ln(x + 1)$. Find the limits as $x\to\infty$ of f and g. Can we apply the Difference Rule to $\lim_{x\to\infty} (f(x) - g(x))$? Explain. Does the limit of the difference exist?
4. Based on parts 1–3, what advice might you give about applying Theorem 5?

Infinite Limits as $x \to a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a, we say that $\lim_{x \to a} f(x) = \infty$. If the values of f become large and negative, exceeding all negative bounds as $x \to a$, we say that $\lim_{x \to a} f(x) = -\infty$.

Looking at $f(x) = 1/x$ (Figure 2.9, page 70), we observe that

$$\lim_{x \to 0^+} 1/x = \infty \quad \text{and} \quad \lim_{x \to 0^-} 1/x = -\infty.$$

We say that the line $x = 0$ is a *vertical asymptote* of the graph of f.

DEFINITION **Vertical Asymptote**

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

EXAMPLE 4 Finding Vertical Asymptotes

Find the vertical asymptotes of $f(x) = \frac{1}{x^2}$. Describe the behavior to the left and right of each vertical asymptote.

$$\lim_{x \to 0^-} \frac{1}{x^2} = \infty$$

vertical asymptote

$x = 0$

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty$$

end behavior

left:

$$\lim_{x \to -\infty} \frac{1}{x^2} = 0$$

right:

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

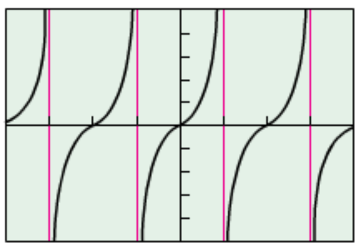
EXAMPLE 5 Finding Vertical Asymptotes

The graph of $f(x) = \tan x = (\sin x)/(\cos x)$ has infinitely many vertical asymptotes, one at each point where the cosine is zero. If a is an odd multiple of $\pi/2$, then

$$\lim_{x \to a^+} \tan x = -\infty \quad \text{and} \quad \lim_{x \to a^-} \tan x = \infty,$$

as suggested by Figur

Now try Exercise 31.



$[-2\pi, 2\pi]$ by $[-5, 5]$

Figure 2.12 The graph of $f(x) = \tan x$ has a vertical asymptote at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. (Example 5)

You might think that the graph of a quotient always has a vertical asymptote where the denominator is zero, but that need not be the case. For example, we observed in Section 2.1 that $\lim_{x \to 0} (\sin x)/x = 1$.

EXAMPLE 6 Modeling Functions For $|x|$ Large

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that while f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ large.

DEFINITION **End Behavior Model**

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$

(b) a **left end behavior model** for f if and only if $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 1.$

EXAMPLE 7 Finding End Behavior Models

Find an end behavior model for

(a) $f(x) = \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$

(b) $g(x) = \frac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

top bigger than bottom

$\lim_{x \to \infty} f(x) \to \infty$

degree same

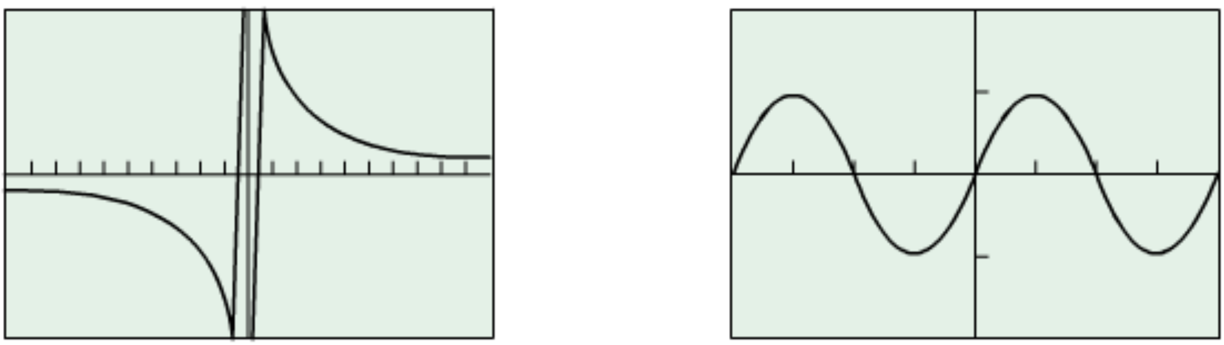
$\lim_{x \to \infty} g(x) = \frac{2}{5}$

EXAMPLE 8 **Finding End Behavior Models**

Let $f(x) = x + e^{-x}$. Show that $g(x) = x$ is a right end behavior model for f while $h(x) = e^{-x}$ is a left end behavior model for f.

EXAMPLE 9 Using Substitution

Find $\lim_{x\to\infty} \sin(1/x)$.



[–10, 10] by [–1, 1]

(a)

[-2π, 2π] by [–2, 2]

(b)

Figure 2.15 The graphs of (a) $f(x) = \sin(1/x)$ and (b) $g(x) = f(1/x) = \sin x$. (Example 9)